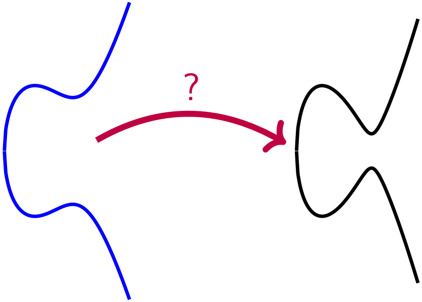
The endomorphism ring problem given an endomorphism

Arthur Herlédan Le Merdy, Benjamin Wesolowski

Tuesday 9$^{\text{th}}$ April, 2024

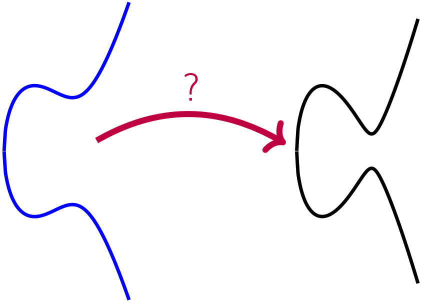
Hard problems

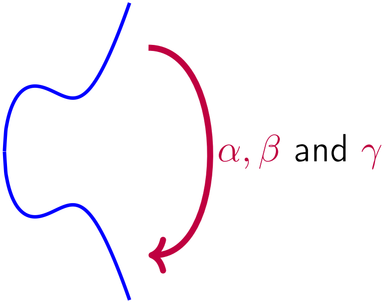
Isogeny Problem



Hard problems

Isogeny Problem $\iff$ Endomorphism Ring Problem





$$\mathrm{End}(\quad) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z} + \gamma\mathbb{Z}$$

Supersingular endomorphism ring problem

The supersingular endomorphism ring problem (EndRing):

Given a **supersingular** elliptic curve $\boldsymbol{E}$, find a basis of its endomorphism ring $\mathbf{End}(\boldsymbol{E})$.

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[Wes21] `EndRing` $\iff$ `Isogeny Problem` **under the Generalized Riemann Hypothesis**.

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[Wes21] `EndRing` $\iff$ `Isogeny Problem` **under the Generalized Riemann Hypothesis**.

- Some protocols give **a public endomorphism** $\theta \in \mathrm{End}(E) \backslash \mathbb{Z}$:

  [Cas+18] CSIDH The Frobenius endomorphism $\pi_E : (x, y) \mapsto (x^p, y^p)$.

  [Feo+23] SCALLOP An $(\mathbb{Z} + f\mathbb{Z}[i])$-orientation with f a large prime.

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	EndRing	EndRing given one endomorphism θ
Classical	$p^{1/2}$	$\exp(\log \deg \theta)$ under heuristics
Quantum	$p^{1/4}$	$\mathrm{subexp}(\log \deg \theta)$ under heuristics

Complexity of EndRing and its variant for an elliptic curve defined over $\mathbb{F}_{p^2}$, with p a prime. [Arp+22]

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Orientations [CK20]

Let $\theta \in \mathrm{End}(E) \backslash \mathbb{Z}$.

- $\mathbb{Z}[\theta] \simeq \mathbb{Z}[X]/ < X^2 + (\hat{\theta} + \theta)X + \deg\theta >$, i.e. $\mathbb{Z}[\theta]$ is a **quadratic order**.
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Let $\mathfrak{O}$ be an order of an imaginary quadratic field K.

- An embedding $\iota : K \hookrightarrow \mathrm{End}(E) \otimes \mathbb{Q}$ is called an ***K*-orientation**, it is an **$\mathfrak{O}$-orientation** if $\iota(\mathfrak{O}) \subseteq \mathrm{End}(E)$.
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Knowing an endomorphism $\longleftrightarrow$ Knowing an orientation

Oriented isogenies

Let $\iota : \mathbb{Z}[\omega] \hookrightarrow \mathrm{End}(E_1)$ be an orientation with $\iota(\omega) = \theta$. Let $\varphi : E_1 \to E_2$ an isogeny.

$$\begin{tikzcd} E_1 \arrow[loop left, "\theta"] \arrow[rr, bend left, "\varphi"] & & E_2 \end{tikzcd}$$

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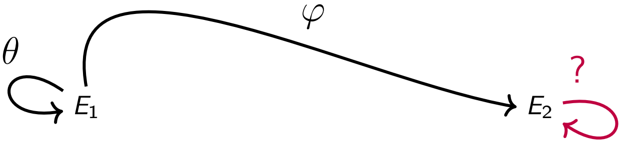

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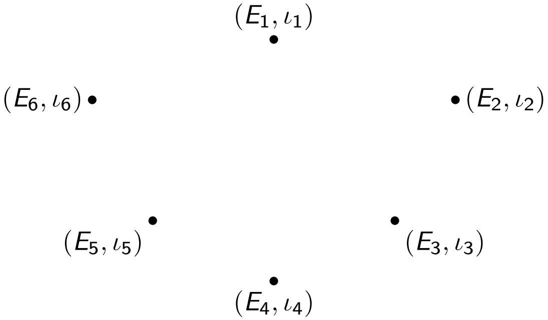
If $\varphi_*(\iota)$ is a primitive $\mathbb{Z}[\omega]$-orientation, then $\varphi : (E_1, \iota) \to (E_2, \varphi_*(\iota))$ is **horizontal**.

Class group action

Let $SS_{\mathfrak{O}}$ be the set of primitive $\mathfrak{O}$-oriented elliptic curves up to isomorphisms.



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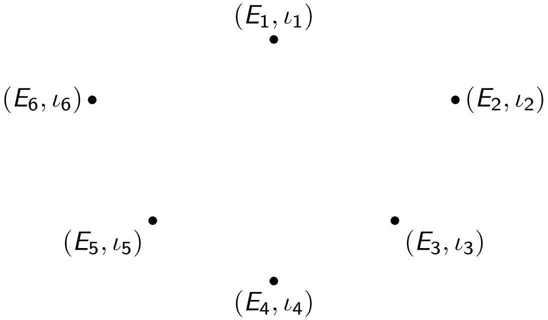

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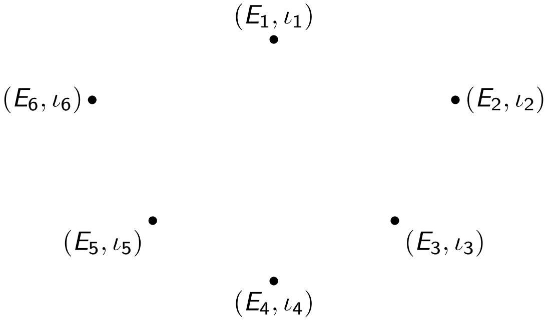

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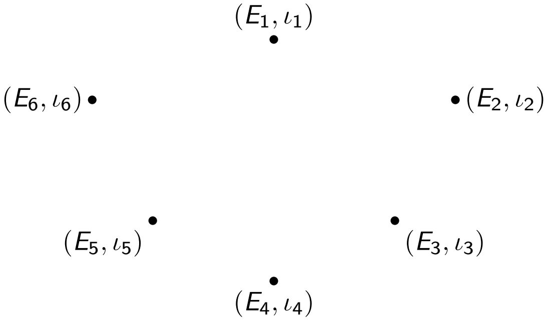

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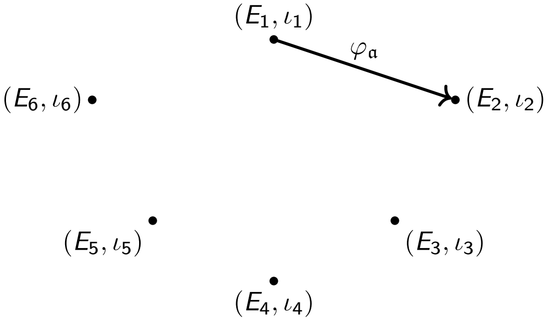

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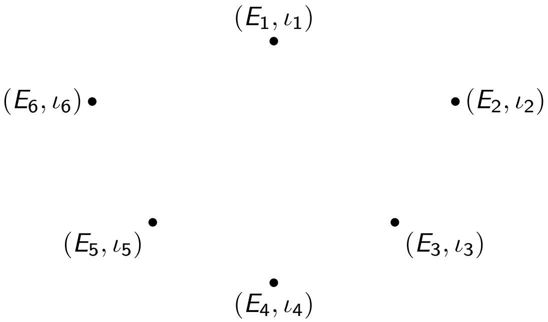

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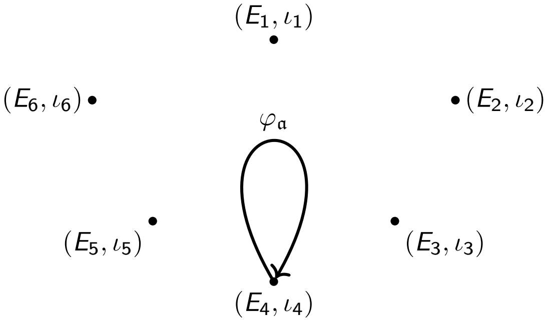

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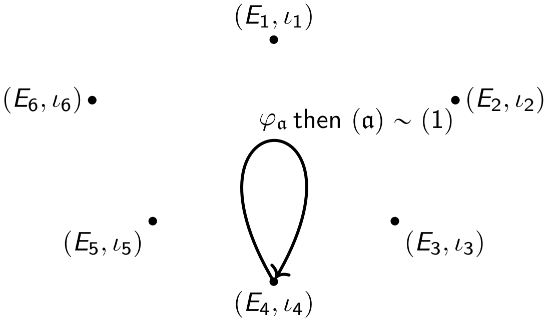

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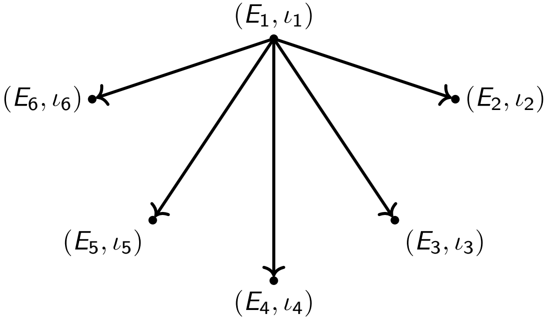

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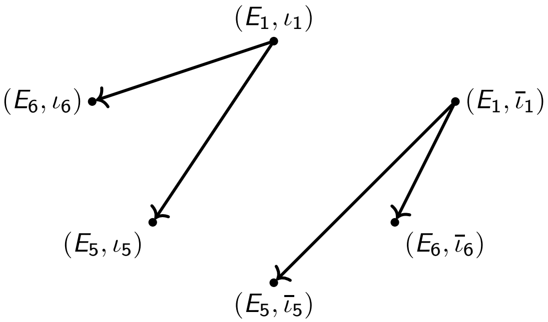

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(E_1, ι_1) $\bullet$

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$$\begin{tikzcd} \text{Knowing an endomorphism} \arrow[r, Leftrightarrow] & \text{Knowing an orientation} \\ & \text{Knowing a primitive orientation} \arrow[u, Rightarrow] \end{tikzcd}$$

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[Wes22] $\Big\downarrow$ **GRH**

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$\mathfrak{O}$-EndRing + factorisation of $\mathrm{disc}(\mathfrak{O})$

$\mathrm{disc}(\mathbb{Z}[\omega]) = (\bar{\omega} - \omega)^2$

[Wes22] **GRH**

$\mathfrak{O}$-Vectorisation

Proof outline

$$\begin{array}{c} \boxed{E \text{ and } \theta \in \mathrm{End}(E)\backslash\mathbb{Z}} \\ \downarrow \\ \boxed{\text{A basis of } \mathrm{End}(E)} \end{array}$$

➜ **Goal:** Give a complexity analysis of the `EndRing` problem given one endomorphism **under GRH only**.

Proof outline

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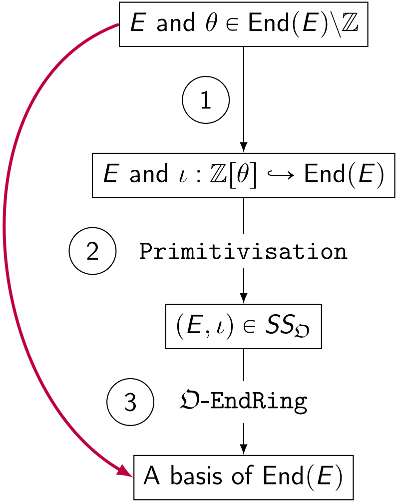

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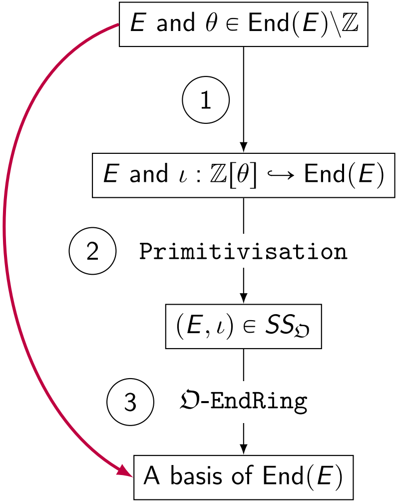

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③ Under some heuristics, classically in $|\mathrm{disc}(\mathfrak{O})|^{1/4}$ and quantumly in $\mathrm{subexp}(\log|\mathrm{disc}(\mathfrak{O})|)$. [Wes22]

Primitivisation problem [Arp+22]

`Primitivisation`:

Given $\boldsymbol{E}$ supersingular and $\boldsymbol{\theta} \in \mathrm{End}(E)$, find the **quadratic order** $\mathfrak{O} := \mathbb{Q}(\theta) \bigcap \mathrm{End}(E)$.

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After SIDH's attacks, **checking an inclusion** is made by **dividing the Frobenius**. [Rob22b].

Division of endomorphism

(Higher dimensional) Interpolation [Rob22a]

Given coprime integers $\boldsymbol{N} < \boldsymbol{N'}$ and four points $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{Q}_1, \boldsymbol{Q}_2$ such that

$$\langle P_1, P_2 \rangle = E_1[N'] \text{ and } \langle Q_1, Q_2 \rangle = E_2[N'] \text{ with } N' \text{ a } B\text{-powersmooth integer.}$$

One can **check the existence** and **compute** in **poly** $(\boldsymbol{l}, \boldsymbol{B})$ time the isogeny of degree N

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Given an endomorphism $\boldsymbol{\theta} \in \mathrm{End}(E)$ and an integer $\boldsymbol{n}$ such that $gcd(\deg \theta, np) = 1$, one can **check if** $\boldsymbol{\theta}/\boldsymbol{n} \in$ **End** $(\boldsymbol{E})$ and **compute it** in **poly** $(\boldsymbol{l})$ time.

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3. Use the higher dimensional interpolation.

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Proposition: `Primitivisation`

Given a $\mathbb{Z}[\omega]$-oriented elliptic curve (E, ι) and **the factorisation of disc** $(\mathbb{Z}[\omega])$, one can compute the **primitive orientation** $\iota' : \mathfrak{O} \hookrightarrow \mathrm{End}(E)$ such that $\mathfrak{O} \supseteq \mathbb{Z}[\omega]$ in **poly** (I).

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Thus successive divisions of $\iota(f\sqrt{\Delta})$ by the prime factors of f gives $\iota' : \mathfrak{O} \hookrightarrow \mathrm{End}(E)$.

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Remark [PR23], `CLAPOTI`: CLass group Action in POlynomial TIme!

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Solving $\mathfrak{O}$-Vect classically

Proposition: Classical $\mathfrak{O}$-`Vectorisation` **(GRH)**

Given $(\boldsymbol{E}_1, \boldsymbol{\iota}_1)$ and $(\boldsymbol{E}_2, \boldsymbol{\iota}_2)$ in $SS_{\mathfrak{O}}$, one can find an $\mathfrak{O}$-ideal $\mathfrak{a}$ such that $\varphi_{\mathfrak{a}} : (E_1, \iota_1) \to (E_2, \iota_2)$ in $\mathbf{poly}\,(\boldsymbol{l}) \cdot |\,\mathbf{disc}\,(\mathfrak{O})|^{1/4}$.

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EndRing given one endomorphism

Theorem: Classical EndRing problem given one endomorphism **(GRH)**

Given an elliptic curve $\boldsymbol{E}$ and an endomorphism $\boldsymbol{\theta} \in \mathrm{End}(E)\backslash\mathbb{Z}$, one can compute a basis of the endomorphism ring $\mathbf{End}(\boldsymbol{E})$ in time $\mathbf{poly}(\boldsymbol{l}) \cdot |\mathbf{disc}(\mathbb{Z}[\boldsymbol{\theta}])|^{1/4}$.

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Thanks for your attention!

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